Relations

Examples

- "Being a sibling of" is a relation. e.g. "Alice is a sibling of Bob"
- "$<$" is a relation between real #s : $3 < 5$, $\pi < 4$
- Being a higher frequency is a relation between pitches

More precisely, a <u>relation</u> on a set A is a subset of $A \times A$.
If C is a relation on A, and $(x,y) \in C$, we can use the notation $x\,C\,y$. (Note: can also have a relation in $X \times Y$)

Ex:

- Let P = set of people in the world. Define the following relations
  - $D = \{(x,y) \mid x \text{ is a descendant of } y\}$
  - $B = \{(x,y) \mid x \text{ and } y \text{ have a common ancestor}\}$
  - $S = \{(x,y) \mid x \text{ and } y \text{ have the same parents}\}$

  Notice: B is <u>symmetric</u> — If $(x,y) \in B$ (or xBy) then $(y,x) \in B$.
  However, D is not!

- $(2,3) \in <$ means $2 < 3$.

Equivalence relations and partitions

An <u>equivalence relation</u> on a set A is a relation C such that

① $x \, C \, x \quad \forall \, x \in A$ (reflexivity) [$<$ is not reflexive]

② If $x \, C \, y$ then $y \, C \, x$ (symmetry) [being a parent of is not symmetric]

③ If $x \, C \, y$ and $y \, C \, z$ then $x \, C \, z$ (transitivity) [being related to is not transitive]

Note: Frequently "$\sim$" is used to denote an equivalence relation.

Given an equivalence relation $\sim$ on A and $x \in A$, define

the equivalence class determined by x to be the subset of A

$$\{ y \mid y \sim x \}$$

(notice that this subset contains x since $x \sim x$)

Examples:

- equality is an equivalence relation. The equivalence class determined by x is $\{x\}$

- having the same rank is an equivalence relation on playing cards
  The equiv. class determined by the 7 of ♥ is
  $\{7♥, 7♦, 7♣, 7♠\}$

- $A \times A$ is an equivalence relation on A ($x \sim y \quad \forall \, x, y \in A$)
  The equiv. class determined by any $x \in A$ is A itself.

Claim: Two equivalence classes E and F are either disjoint or equal.

Proof: Let E be determined by x, F by y.

Suppose $E \cap F$ is not empty. Let $z \in E \cap F$.

Then $z \sim x$ and $z \sim y$. Symmetry implies

$x \sim z$ and $z \sim y$, and it follows from transitivity that

$x \sim y$.

We want to show $E = F$. If $a \in E$, then $a \sim x$ by definition.

Again by transitivity, $a \sim y$, so $a \in F$. Thus $E \subseteq F$.

By an analogous argument, $F \subseteq E$, so $E = F$. □

(Note: we wanted to show $P \vee Q$. Instead we showed $(\neg P) \Rightarrow Q$.

Do you see why $((\neg P) \Rightarrow Q) \Leftrightarrow (P \vee Q)$?)

Let $\underset{=}{\overset{A/\sim}{\mathcal{E}}}$ be the set of equivalence classes corresponding to an equiv. relation $\sim$. Then distinct elements of $\mathcal{E}$ are disjoint (i.e. $E \neq F \Rightarrow E \cap F = \emptyset$) and $\bigcup_{S \in \mathcal{E}} S = A$. More generally, this is called a partition of A:

Def: If A is a set, a partition of A is a collection of disjoint nonempty subsets of A whose union is A.

Let $\mathcal{C}$ be a partition of A. Then we can define an equivalence relation $\sim$ by $x \sim y \Longleftrightarrow x, y \in X$ for some $X \in \mathcal{C}$.

This is called the relation induced by $\mathcal{C}$

In this case, $A/\sim \; = \mathcal{C}$.

Examples

1.) Define $P, Q \in \mathbb{R}^2$ to be equivalent if they are the same distance from the origin. Then $\mathbb{R}^2/\sim$ is the set of circles centered at the origin (along with the set consisting only of the origin).

2.) Define $P \sim Q$ in $\mathbb{R}^2$ if P and Q have the same x-coordinate. Then $\mathbb{R}^2/\sim$ is the set of all vertical lines.

If R is any relation, then the <u>domain</u> of R, is

$$\operatorname{dom} R = \{x \mid xRy \text{ for some } y\} \text{ and}$$

the <u>range</u> of R is $\operatorname{ran} R = \{y \mid xRy \text{ for some } x\}$

If R is an equivalence relation on A, then $\operatorname{dom} R = \operatorname{ran} R = A$, since $xRx \;\; \forall \; x \in A$.